3961. Proposed by Michel Bataille.

In a triangle ABC, let $\angle A \geq \angle B \geq \angle C$ and suppose that

$$\sin 4A + \sin 4B + \sin 4C = 2(\sin 2A + \sin 2B + \sin 2C).$$

Find all possible values of $\cos A$.

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Let's move on to the more compact notation denoting $\alpha := \angle A, \beta := \angle B, \gamma := \angle C$.

Then $\alpha \geq \beta \geq \gamma > 0, \alpha + \beta + \gamma = \pi$ and

(**1**) $\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$

$\Leftrightarrow 4\sin 2(\alpha+\beta)\sin 2(\beta+\gamma)\sin 2(\gamma+\beta) =$

$8\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\beta) \Leftrightarrow -\sin 2\gamma \sin 2\alpha \sin 2\beta = 2\sin\gamma\sin\alpha\sin\beta \Leftrightarrow$

$-8\sin\alpha\cos\alpha\sin\beta\cos\beta\sin\gamma\cos\gamma = 2\sin\gamma\sin\alpha\sin\beta \Leftrightarrow \sin\gamma\sin\alpha\sin\beta(1+4\cos\alpha\cos\beta\cos\gamma) =$

$1+4\cos\alpha\cos\beta\cos\gamma = 0$ because $\sin\gamma\sin\alpha\sin\beta > 0.$

Let $t := -\cos\alpha$. Then we have

$2\cos\beta\cos\gamma = \cos(\beta+\gamma)+\cos(\beta-\gamma) = \cos(\beta-\gamma)-\cos\alpha = \cos(\beta-\gamma)+t$ and

$1+4\cos\alpha\cos\beta\cos\gamma = 0$ becomes $1-2t(\cos(\beta-\gamma)+t) = 0 \Leftrightarrow \cos(\beta-\gamma) = \dfrac{1}{2t} - t.$

Since $4\cos\alpha\cos\beta\cos\gamma = -1 < 0$ and $\alpha \geq \beta \geq \gamma > 0$ and $\alpha+\beta+\gamma = \pi$ then

$\alpha > \pi/2 \Leftrightarrow t > 0$ and $0 < \gamma \leq \beta < \pi/2.$

Therefore, $0 < \cos(\beta-\gamma) \leq 1 \Leftrightarrow 0 < \dfrac{1}{2t} - t \leq 1 \Leftrightarrow \dfrac{\sqrt{3}-1}{2} \leq t < \dfrac{1}{\sqrt{2}}.$

And on the contrary, for $\dfrac{\sqrt{3}-1}{2} \leq t < \dfrac{1}{\sqrt{2}}$ we have $1+4\cos\alpha\cos\beta\cos\gamma = 0 \Leftrightarrow$

$$\begin{cases} \alpha = \cos^{-1}(-t) \\ \beta-\gamma = \cos^{-1}\left(\frac{1}{2t}-t\right) \in \left(0, \frac{\pi}{2}\right) \\ \beta+\gamma = \pi-\alpha = \cos^{-1}(t) \in \left(0, \frac{\pi}{2}\right) \end{cases} \Rightarrow \alpha \geq \beta \geq \gamma > 0, \alpha+\beta+\gamma = \pi \text{ and } (\mathbf{1}).$$

Thus, all possible values of $\cos\alpha$ represented by set $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{\sqrt{3}-1}{2}\right].$